MATH 2010E Summer 2021-22
Final Examination, June 29, 2022
Time: 1:30 PM-3:15 PM

- Make your solution into a single PDF file and upload on Blackboard by 3:35 PM.
- There are 8 problems on this exam, worth 100 points in total.
- The Final Exam is open-book/notes. You are allowed to make use of the internet during the exam, except for the purpose of communicating with another individual besides the lecturer and TA of this course. Calculators are allowed.
- You are not allowed to post questions or answers on any internet forum or tutoring / educational establishment during the exam. Any violation of this rule is considered an act of academic misconduct, and will be reported to the Department for further disciplinary review.
- All work submitted should be your own. Students found to have highly similar answers could be subjected to disciplinary review. Please kindly be reminded of the following regulations enforced by the university:

  Honesty in Academic Work: The Chinese University of Hong Kong places very high importance on honesty in academic work submitted by students, and adopts a policy of zero tolerance on cheating and plagiarism. Any related offence will lead to disciplinary action including termination of studies at the University.

Unless otherwise noted, please justify all your answers.

1. (10 pts) Let

$$f(x,y) = \begin{cases} \dfrac{xy}{|x|+|y|} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Determine if f is differentiable at $(0,0)$.

2. (10 pts) Let $f(x,y) = x^2 + xy + y^2$.

   (a) (2 pts) Compute ∇f at $(2,-1)$.

   (b) (6 pts) What is the equation of the tangent plane to the graph of f at $(2,-1,3)$?

   (c) (2 pts) Find the direction which f decreases most rapidly at $(2,-1)$. What is the rate of change of f in that direction?

3. (16 pts) For (a) and (b) in this problem, you are not required to justify your answer. You need to justify (c).

   (a) (3 pts, only answer) Find the second-order Taylor polynomial of $f(x,y) = 1 + 2x + 3y + 4x^2 + 5y^3 + 6xy^3$ at $(0,0)$.

   (b) (3 pts, only answer) Let $f(x,y)$ be as in (a). Find the second-order Taylor polynomial of $g(x,y) = f(x,y)\cos(2(x+y))$ at $(0,0)$.

   (c) (10 pts) Let $\vec{v} = (a,b)$ and $g(x,y)$ be as in (b). Express $D_{\vec{v}}^2 g$, which is by definition $D_{\vec{v}}(D_{\vec{v}} g)$, using a and b.

4. (12 pts) Let

$$f(x,y) = 2x^3 + y^2 - 4xy + x^2 - 24x + 6y + 7.$$

Find all critical points of f and determine whether they are local maxima, local minima, or saddle points.

5. (12 pts) Find the global maximum and minimum of

$$f(x,y,z) = (2x - y + 3z)e^{\frac{x^2+y^2+z^2}{2}}$$

on $A = \{(x,y,z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 \leq 14\}$.

6. (12 pts) Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $G(x,y) = (x+y, xy)$. $F : \mathbb{R}^2 \to \mathbb{R}^2$ be differentiable function such that $F(3,2) = (1,0)$. Suppose F and G satisfy

$$(G \circ F)(t,2) = (t^2, 1+t), (G \circ F)(3,s) = (1, s^2).$$

Compute the Jacobian matrix $DF(3,2)$.

7. (16 pts) Let C be a curve in $\mathbb{R}^3$ defined by the intersection of the following two surfaces:

$$S_1 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 9\}$$

$$S_2 = \{(x, y, z) \in \mathbb{R}^3 | z^3 - 3xy + y^2 = 6\}.$$

(a) (8 pts) Show that y and z can be solved as a fuction of x near the point $(1, 2, 2)$. Also, compute $\dfrac{\partial y}{\partial x}$ and $\dfrac{\partial z}{\partial x}$ at $(1, 2, 2)$.

(b) (8 pts) Give a parametrization of the tangent line of C at $(1, 2, 2)$.

8. (12 pts) Let $F : \mathbb{R}^2 - \{(0, 0)\} \to \mathbb{R}^2$ be defined by

$$F(x, y) = (\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}).$$

(a) (6 pts) Show that F is invertible near any point.

(b) (6 pts) Let G be a local inverse of F near $(1, 1)$. Compute the Jacobian matrix $DG(\frac{1}{2}, \frac{1}{2})$.